AD628625

MEMORANDUM
RM-4657-PR
FEBRUARY 1966

ON THE CONTINUITY OF THE MINIMUM SET OF A CONTINUOUS FUNCTION

George B. Dantzig, Jon H. Folkman and Norman Shapiro

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UNITED STATES AIR FORCE PROJECT RAND

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This research is sponsored by the United States Air Force under Project RAND—Contract No. AF 49(638)-1700—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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PREFACE

This Memorandum reports on recent work which is part of a continuing RAND Corporation study of the mathematical and computational aspects of chemical equilibrium theory (see, for example, [4-13]). It may also be regarded as representing part of RAND's work in mathematical programming (see bibliography in [3]) because it deals with a subject of more general mathematical interest than many of the previous RAND publications in this series.

RAND's research in chemical equilibrium theory is complementary to its continuing research on the application of mathematics and computer technology to biochemistry and human physiology [14-21], and is also applicable to, for example, studies of the atmospheres of the earth and other planets, to the computation of the characteristics of propulsive fluids, and to the examination of certain reentry problems.

ABSTRACT

This Memorandum obtains necessary and sufficient conditions so that the solution of a constrained minimization problem will vary continuously when the constraints and objective function are varied. It also obtains special results in the case that the constraints are linear inequalities.

CONTENTS

ON THE CONTINUITY OF THE MINIMUM SET OF A CONTINUOUS FUNCTION

In this Memorandum we are concerned with the question: How does the solution of a constrained minimization problem vary when the constraints and/or the objective function are varied? In particular we ask for conditions on the constraints, the objective function, the manner in which the constraints are varied, and the manner in which the objective function is varied, for the minimum (or more precisely, the point at which the minimum is attained) to vary in some sort of "continuous" manner.

One difficulty is that the minimum may not be unique. We resolve this difficulty in two different ways: (1) By dealing with the set of minima and using an appropriate notion for the continuous variation of a set. (2) By finding conditions for the continuous variation of the minimum, whenever it is unique.

It turns out that the variation of the objective function offers little difficulty and can be dealt with easily in two theorems. Thus, we concentrate largely on the dependence of the set of minima (or of the unique minimum) on the constraint set.

In Part I we deal with the general problem. In Part II we deal with the special case that the constraint set is defined by linear inequalities and equations. That is, we ask for conditions that the set of points, x, which minimize $f(x, p)$ subject to the constraint that $ax \leq b$, is a "continuous" function of the parameters p, the matrix a, and the vector b. Here f is a real-valued function defined on $E^n \times P$, where P is the space in which p varies.

I

THE GENERAL PROBLEM

Section I.1

Let $\{A_n\}$ be a sequence of subsets of a metric space X. We define the outer limit, $\overline{\lim}_{n\to\infty} A_n$, by

$$\overline{\lim_{n\to\infty}} A_n = \{x \in X \mid x = \lim_{i\to\infty} x_{n_i}, \text{ where } \{n_i\} \text{ is an infinite subsequence of the integers and } x_{n_i} \in A_{n_i}\}.$$

We define the inner limit, $\underline{\lim}_{n\to\infty} A_n$, by

$$\underline{\lim_{n\to\infty}} A_n = \{x \in X \mid x = \lim_{n\to\infty} x_n, \text{ where } x_n \in A_n \text{ for all but a finite number of } n\}.$$

If $\overline{\lim}_{n\to\infty} A_n = \underline{\lim}_{n\to\infty} A_n$ we say that the limit, $\lim_{n\to\infty} A_n$, exists and set

$$\lim_{n\to\infty} A_n = \overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n.$$

If Y is any subset of X, then $\overline{Y}$ is the topological closure of Y and Bndry Y is the boundary of Y.

The following properties follow immediately from the definitions. We let I denote the set of all infinite subsequences $\{n_i\}$ of the positive integers.

(I.1.1) $\overline{\lim_{n\to\infty}} A_n = \bigcup_{\{n_i\}\in I} \underline{\lim_{i\to\infty}} A_{n_i}$.

(I.1.2) $\underline{\lim_{n\to\infty}} A_n = \bigcap_{\{n_i\}\in I} \overline{\lim_{i\to\infty}} A_{n_i}$.

(I.1.3) $\overline{\lim_{n\to\infty}} A_n = \bigcap_{m=1}^{\infty} \left(\overline{\bigcup_{n=m}^{\infty} A_n}\right)$.

(I.1.4) $\overline{\lim_{n\to\infty}} A_n$ and $\underline{\lim_{n\to\infty}} A_n$ are closed.

(I.1.5) If $A_n \subset B_n$ for all sufficiently large n, then $\overline{\lim_{n\to\infty}} A_n \subset \overline{\lim_{n\to\infty}} B_n$ and $\underline{\lim_{n\to\infty}} A_n \subset \underline{\lim_{n\to\infty}} B_n$.

(I.1.6) If $A_n = A$ for all sufficiently large n, then $\lim_{n\to\infty} A_n$ exists and is equal to $\bar{A}$.

(I.1.7) If A_n is a sequence of convex subsets of E^n, then $\underline{\lim_{n\to\infty}} A_n$ is convex.

Suppose that F is a set-valued function from the metric space X to the metric space Y. That is, suppose the domain of F is X and that the range of F consists of subsets of Y. We define F^* to be the function which is defined for those $x \in X$ for which $F(x)$ contains exactly one element, and $F^*(x)$ is defined to be that element. Following Berge ([1], p. 111) we say that F is <u>closed at the point</u> $x_0 \in X$ if, for all sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ satisfying

$x_n \to x_0$, $y_n \to y_0$, $y_n \in F(x_n)$, we have $y_0 \in F(x_0)$. With the above terminology this becomes: For all sequences $\{x_n\} \subset X$ with $x_n \to x_0$, we have $\underline{\lim}_{n\to\infty} F(x_n) \subset F(x_0)$. We say that F is closed if it is closed at each point $x \in X$.

Note that the associated single-valued function F^* need not be continuous even though F is closed. For example, the set-valued function from the reals to the reals given by

$$F(x) = \begin{cases} \{\frac{1}{x}\} & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0. \end{cases}$$

is closed, but F^* is discontinuous at zero.

If F is closed, then F^* is continuous if and only if F^* takes compact sets onto compact sets.

If φ is a real valued function on a metric space X and H is a subset of X, we define $M(\varphi|H)$ to be the subset of H where φ achieves its minimum. More precisely,

$$M(\varphi|H) = \{x \in H \mid \varphi(x) = \inf\{\varphi(y) \mid y \in H\}\}.$$

Note that $M(\varphi|H)$ may be thought of as the set of solutions of a constrained minimization problem. If H is a set-valued function from a metric space T to X, then $M(\varphi|H(t))$ defines a set-valued function from T to

X. We will be interested in finding conditions for this function to be a closed function of t and for the associated point-valued function $M^*(\varphi|H(t))$ to be continuous.

The remainder of this section is devoted to several results which we will find useful in the sequel. Most readers will prefer to defer them until completing the rest of the paper.

<u>Theorem I.1.8</u>. Let X be a metric space and let $\{A_n\}$ be a sequence of connected subsets of X. Let U be an open subset of X with compact boundary. If $\underline{\lim}_{n\to\infty} A_n$ is nonempty and $\overline{\lim}_{n\to\infty} A_n \subset U$, then $A_n \subset U$ for all sufficiently large n.

<u>Proof</u>. Let $x \in \underline{\lim}_{n\to\infty} A_n \subset \overline{\lim}_{n\to\infty} A_n \subset U$. There is a sequence $\{x_n\}$ with $x_n \to x$ and $x_n \in A_n$ for n large. Hence, $x_n \in A_n \cap U$ for n large. Suppose the conclusion does not hold. Then we can find a sequence $\{n_i\}$ and points $y_i \in A_{n_i}$ such that $y_i \notin U$.

For i sufficiently large, $x_{n_i} \in A_{n_i} \cap U$ and $y_i \in A_{n_i} - U$. Hence, since A_{n_i} is connected, there is a point $z_i \in A_{n_i} \cap \text{Bndry } U$. Now Bndry U is compact, so a subsequence of the z_i converges to a point $z \in \text{Bndry } U$. But $z \in \overline{\lim}_{n\to\infty} A_n \subset U$. <u>Q.E.D</u>.

<u>Theorem I.1.9</u>. Let X be a locally compact metric space and let $\{A_n\}$ be a sequence of connected subsets of X. If $\underline{\lim}_{n\to\infty} A_n$ is nonempty and $\overline{\lim}_{n\to\infty} A_n$ is compact, then $\overline{\lim}_{n\to\infty} A_n$ is connected.

<u>Proof</u>. Suppose not. Then there are disjoint open sets $U, V \subset X$ such that $\overline{\lim}_{n\to\infty} A_n \subset U \cup V$, and $U \cap \overline{\lim}_{n\to\infty} A_n$ and $V \cap \overline{\lim}_{n\to\infty} A_n$ are both nonempty. Since $\overline{\lim}_{n\to\infty} A_n$ is compact and X is locally compact, we may choose U and V so that $\overline{U \cup V}$ is compact. Hence, Bndry $(U \cup V)$ is compact. By Theorem I.1.8, $A_n \subset U \cup V$ for n sufficiently large.

Let $x \in \underline{\lim}_{n\to\infty} A_n$ and let $x_n \to x$ with $x_n \in A_n$ for n sufficiently large. We have $x \in \overline{\lim}_{n\to\infty} A_n \subset U \cup V$, so we may assume $x \in U$. Then, for n sufficiently large,

$A_n \subset U \cup V$ and $x_n \in A_n \cap U$. Since A_n is connected, $A_n \subset U$ for n sufficiently large. Hence, $\overline{\lim\limits_{n\to\infty}} A_n \subset \bar{U}$. But $\bar{U} \cap V = \varphi$, so this contradicts the assumption that $V \cap \overline{\lim\limits_{n\to\infty}} A_n$ is nonempty. Q.E.D.

Theorem I.1.10. Let $\{A_n\}$ be a sequence of subsets of the metric space X. Let U be an open subset of X and suppose that $\left(\underline{\lim\limits_{n\to\infty}} A_n\right) \cap \bar{U} \subset \overline{\left(\underline{\lim\limits_{n\to\infty}} A_n\right) \cap U}$. Then $\underline{\lim\limits_{n\to\infty}} (A_n \cap \bar{U}) = \left(\underline{\lim\limits_{n\to\infty}} A_n\right) \cap \bar{U}$.

Proof. $A_n \cap \bar{U} \subset A_n$ and $A_n \cap \bar{U} \subset \bar{U}$, so by (I.1.5) and (I.1.6), $\underline{\lim\limits_{n\to\infty}} (A_n \cap \bar{U}) \subset \left(\underline{\lim\limits_{n\to\infty}} A_n\right) \cap \bar{U}$. By assumption, $\left(\underline{\lim\limits_{n\to\infty}} A_n\right) \cap \bar{U} \subset \overline{\left(\underline{\lim\limits_{n\to\infty}} A_n\right) \cap U}$. Let $x \in (\underline{\lim} A_n) \cap U$. Then $x = \lim\limits_{n\to\infty} x_n$, where $x_n \in A_n$ for n sufficiently large and $x \in U$. Therefore, $x_n \in A_n \cap U \subset A_n \cap \bar{U}$ for n sufficiently large, so $x \in \underline{\lim\limits_{n\to\infty}} (A_n \cap \bar{U})$. This set is closed, so $\overline{\left(\underline{\lim\limits_{n\to\infty}} A_n \cap U\right)} \subset \underline{\lim\limits_{n\to\infty}} (A_n \cap \bar{U})$. Q.E.D.

Theorem I.1.11. Let y be a point in E^m and let $\{H_n\}$ be a sequence of nonempty closed subsets of E^m. There is a nonnegative convex function φ defined on E^m such that $M(\varphi | E^m) = \{y\}$ and $M(\varphi | H_n) = \{x_n\}$ for some $x_n \in H_n$. Furthermore, given any such φ, if $y \in \underline{\lim\limits_{n\to\infty}} H_n$ then $y = \lim\limits_{n\to\infty} x_n$.

Proof. We construct a sequence $\{x_n\}$ of points in of functions on E^m such that each sequence is defined for $0 \le n < \infty$, $x_0 = y$, and such that for $n \ge 0$:

(1) $A_n > 0$,

(2) $A_{n+1} \le \frac{A_n}{10}$,

(3) $A_n \|y - x_n\| \le \frac{1}{2^n}$,

(4) $\varphi_n(x) = \sum_{i=0}^{n} A_i \|x - x_i\|$,

(5) $x_{n+1} \in M(\varphi_n | H_{n+1})$,

where $\| \cdot \|$ denotes the usual norm in E^m.

Set $x_0 = y$, $A_0 = 1$, and $\varphi_0(x) = \|x - x_0\|$. Suppose that x_i, A_i, and φ_i are defined for $0 \le i \le n$. Then $M(\varphi_n | H_{n+1})$ is nonempty because H_{n+1} is closed and nonempty, and for any real number b, $\{x \in E^m | \varphi_n(x) \le b\}$ is bounded. Choose $x_{n+1} \in M(\varphi_n | H_{n+1})$. Choose A_{n+1} to be any real number satisfying (1), (2), and (3). Define φ_{n+1} by (4).

We now define

$$\varphi(x) = \lim_{n \to \infty} \varphi_n(x) = \sum_{n=0}^{\infty} A_n \|x - x_n\|.$$

To show that this limit exists it suffices to show that $\lim_{n \to \infty} \sum_{i=n}^{\infty} A_i \|x - x_i\| = 0$. But

$$\sum_{i=n}^{\infty} A_i\|x - x_i\| \leq \Big(\sum_{i=n}^{\infty} A_i\Big) \|x - y\| + \sum_{i=n}^{\infty} A_i\|y - x_i\|$$

which tends to zero as $n \to \infty$ by virtue of (2) and (3). Since φ is the pointwise limit of nonnegative convex functions, it is nonnegative and convex.

Let $H_0 = E^m$ and we will show that φ achieves its minimum on H_n uniquely at x_n for $n \geq 0$. Note that if we set $\varphi_{-1} = 0$, then (5) holds for $n = -1$. Let $x \in H_n$, $x \neq x_n$. Then

$$\begin{aligned}\varphi(x) - \varphi(x_n) &= \varphi_{n-1}(x) - \varphi_{n-1}(x_n) + A_n\|x - x_n\| \\ &\quad + \sum_{i=n+1}^{\infty} A_i(\|x - x_i\| - \|x_n - x_i\|) \\ &\geq \varphi_{n-1}(x) - \varphi_{n-1}(x_n) + A_n\|x - x_n\| \\ &\quad + \sum_{i=n+1}^{\infty} A_i(\|x - x_i\| - \|x_n - x\| - \|x - x_i\|) \\ &= \varphi_{n-1}(x) - \varphi_{n-1}(x_n) + \Big(A_n - \sum_{i=n+1}^{\infty} A_i\Big)\|x_n - x\| \\ &\geq \varphi_{n-1}(x) - \varphi_{n-1}(x_n) + \frac{8}{9} A_n\|x_n - x\|\end{aligned}$$

by (2). Hence, by (5)

$$\varphi(x) - \varphi(x_n) \geq \frac{8}{9} A_n\|x_n - x\| > 0.$$

This proves the first part of the theorem.

Now suppose $y \in \underline{\lim}_{n\to\infty} H_n$. Then there is a sequence $\{y_n\}$ with $y_n \in H_n$ and $y_n \to y$. Since φ is convex on E^m it is continuous, so $\varphi(y_n) \to \varphi(y)$. But $\varphi(y_n) \geq \varphi(x_n) \geq \varphi(y)$, so $\varphi(x_n) \to \varphi(y)$. Let $\epsilon > 0$. Let

$$\delta = \min\{\varphi(x) \mid x \in E^m, \|x - y\| = \epsilon\} - \varphi(y).$$

The minimum exists because φ is continuous and the set $\{x \in E^m \mid \|x - y\| = \epsilon\}$ is compact. Furthermore, $\delta > 0$ because φ achieves its minimum uniquely at y. Let N be so large that $|\varphi(x_n) - \varphi(y)| < \delta$ for $n \geq N$.

Suppose that for some $n \geq N$, $\|x_n - y\| \geq \epsilon$. Then there is a t with $0 < t \leq 1$ such that if $x = tx_n + (1-t)y$, then $\|x - y\| = \epsilon$. Hence,

$$\begin{aligned}\varphi(y) + \delta \leq \varphi(x) \leq t\varphi(x_n) &+ (1-t)\varphi(y) \\ &< t(\varphi(y) + \delta) + (1-t)\varphi(y) \\ &\leq \varphi(y) + \delta.\end{aligned}$$

This is a contradiction. Therefore, $\|x_n - y\| < \epsilon$ for $n \geq N$, so $x_n \to y$. <u>Q.E.D.</u>

Section I.2

In this section we obtain necessary and sufficient conditions on the variation of φ and of H so that $M(\varphi|H)$ varies in a closed manner. We first obtain a necessary condition on the variation of H, assuming φ to be an arbitrary but fixed continuous function.

Theorem I.2.1. Let T and X be metric spaces, and let H be a set-valued function from T to X. Suppose that for every continuous $\varphi: X \to R$ the set-valued function

$$t \to M(\varphi|H(t))$$

is closed at the point $t_0 \in T$. Then, for every sequence $\{t_n\} \subset T$ with $t_n \to t_0$, $\underline{\lim}_{n\to\infty} H(t_n)$ is either empty or equal to $H(t_0)$.

Proof. Let $\{t_n\} \subset T$ be a sequence with $t_n \to t_0$. Suppose $\underline{\lim}_{n\to\infty} H(t_n)$ is nonempty. Then there is an $x_0 \in \underline{\lim}_{n\to\infty} H(t_n)$ and $x_0 = \lim_{n\to\infty} x_n$ where $x_n \in H(t_n)$ for n sufficiently large. Define $\varphi: X \to R$ by

$$\varphi(x) = \inf_n d(x, x_n),$$

where d is the metric in X. We have $\varphi(x) \geq 0$ for all x. $\varphi(x_n) = 0$ and $x_n \in H(t_n)$ for n large, so

$x_n \in M(\varphi|H(t_n))$ for n large. Consequently, since φ is continuous,

$$x_0 = \lim x_n \in \underline{\lim}\, M(\varphi|H(t_n)) \subset M(\varphi|H(t_0)) \subset H(t_0).$$

Therefore, $\underline{\lim}_{n\to\infty} H(t_n) \subset H(t_0)$.

Suppose that $\underline{\lim}_{n\to\infty} H(t_n) \neq H(t_0)$. Then there is a point $x_0' \in H(t_0)$ with $x_0' \notin \underline{\lim}_{n\to\infty} H(t_n)$. Hence, there is an $\epsilon > 0$ and an infinite subsequence $\{n_i\}$ of the positive integers such that

$$d(x_0', H(t_{n_i})) \geq \epsilon \quad \text{for all} \quad i.$$

Define $\psi: X \to R$ by

$$\psi(x) = \min\,(d(x, x_0'), \inf_n d(x, x_n) + \epsilon).$$

For $x \in H(t_{n_i})$, $\psi(x) \geq \epsilon$. Now $\psi(x_{n_i}) = \epsilon$, so $x_{n_i} \in M(\psi|H(t_{n_i}))$. Since $\lim_{i\to\infty} t_{n_i} = t_0$, we have

$$x_0 = \lim_{i\to\infty} x_{n_i} \in \underline{\lim}_{i\to\infty} M(\psi|H(t_{n_i})) \subset M(\psi|H(t_0)).$$

But

$$\psi(x_0) = \lim_{i\to\infty} \psi(x_{n_i}) = \epsilon > 0 = \psi(x_0')$$

and $x_0' \in H(t_0)$, so this is a contradiction. Q.E.D.

We now prove the converse of Theorem I.2.1 in a strengthened form.

Theorem I.2.2. Let H be a set-valued function from the metric space T to the metric space X. Let $t_0 \in T$ and suppose that for every sequence $\{t_n\} \subset T$ with $t_n \to t_0$, $\underline{\lim}_{n\to\infty} H(T_n)$ is either empty or equal to $H(t_0)$. Let $\{\varphi_n\}$ and φ be continuous real-valued functions on X such that $\varphi_n \to \varphi$ uniformly on compact sets. Then, for each sequence $\{t_n\} \subset T$ with $t_n \to t_0$,

$$\underline{\lim}_{n\to\infty} M(\varphi_n | H(t_n)) \subset M(\varphi | H(t_0)).$$

In particular, taking $\varphi_n = \varphi$ for all n, the set-valued function

$$t \to M(\varphi | H(t))$$

is closed at t_0.

Proof. Let $\{t_n\} \subset T$ with $t_n \to t_0$. Suppose $x_0 \in \underline{\lim}_{n\to\infty} M(\varphi_n | H(t_n))$. Then $x_0 = \lim_{n\to\infty} x_n$ where $x_n \in M(\varphi_n | H(t_n))$ for n large. We have $x_0 \in \underline{\lim}_{n\to\infty} H(t_n)$, so $\underline{\lim}_{n\to\infty} H(t_n) = H(t_0)$. Let $x_0' \in H(t_0)$. Then $x_0' = \lim_{n\to\infty} x_n'$, where $x_n' \in H(t_n)$ for n large. Hence, $\varphi_n(x_n) \leq \varphi_n(x_n')$ for n large.

The set consisting of the points $\{x_n\}$, $\{x_n'\}$, x_0, and x_0' is the union of two convergent sequences together with their limits, so it is compact. Now $\varphi_n \to \varphi$ uniformly on compact sets. Consequently,

$$\varphi(x_0) = \lim_{n\to\infty} \varphi_n(x_n) \leq \lim_{n\to\infty} \varphi_n(x_n') = \varphi(x_0').$$

Since this holds for every $x_0' \in H(t_0)$, $x_0 \in M(\varphi | H(t_0))$.
Q.E.D.

Corollary I.2.3. Suppose that the functions $\{\varphi_n\}$ and φ in Theorem I.2.2 have the form

$$\varphi_n(x) = \psi(x, p_n),$$
$$\varphi(x) = \psi(x, p_0),$$

where ψ is a continuous real-valued function on $X \times P$, P is a metric space, $\{p_n\}$ and p_0 are points in P with $p_n \to p_0$. Then

$$\underline{\lim}_{n\to\infty} M(\varphi_n | H(t_n)) \subset M(\varphi | H(t_0)).$$

Proof. It suffices to show that $\varphi_n \to \varphi$ uniformly on compact subsets of X. Let $Q \subset X$ be compact and let

$\epsilon > 0$. For each $x \in Q$ there is a neighborhood U_x of x in X and a neighborhood V_x of p_0 in P such that $|\psi(y, p) - \psi(x, p_0)| < \frac{\epsilon}{2}$ for $y \in U_x$ and $P \in V_x$. The $\{U_x\}_{x \in Q}$ form a covering of Q. Let $U_{x_1}, \ldots, U_{x_n}$ be a finite subcovering. Let $V = V_{x_1} \cap \ldots \cap V_{x_n}$. Now V is a neighborhood of p_0, so there is an N such that $p_n \in V$ for $n \geq N$. Let $x \in Q$ and let $n \geq N$. Then $x \in U_{x_i}$ for some i and $p_n, p_0 \in V \subset V_{x_i}$. Hence,

$$|\varphi_n(x) - \varphi(x)| = |\psi(x, p_n) - \psi(x, p_0)|$$

$$\leq |\psi(x, p_n) - \psi(x_i, p_0)| + |\psi(x_i, p_0) - \psi(x, p_0)| < \epsilon.$$

Q.E.D.

If H is a set-valued function from the metric space T to the metric space X, Theorems I.2.1 and I.2.2 give necessary and sufficient conditions on H for the set-valued function $t \to M(\varphi|H(t))$ to be closed for every continuous $\varphi: X \to R$.

Furthermore, taken together with Corollary I.2.3, they imply, roughly speaking, that if the variation of H is such that $M(\varphi|H)$ varies in a closed manner for each fixed, continuous φ, then $M(\varphi|H)$ varies in a closed manner if φ is allowed to vary by means of a parameter in which it is continuous. The next corollary treats the case of fixed H and variable φ. This result has a simple direct proof and is almost certainly known, although we have not found a reference.

<u>Corollary I.2.4</u>. Let H be a metric space. Let φ_n and φ be continuous real-valued functions on H such that $\varphi_n \to \varphi$ uniformly on compact sets, then

$$\underline{\lim}_{n\to\infty} M(\varphi_n|H) \subset M(\varphi|H).$$

<u>Proof</u>. Clear from Theorem I.2.2. <u>Q.E.D</u>.

It is clear from Theorem I.2.1 and I.2.2 and Corollary I.2.3 that the closed variation of $M(\varphi|H)$ with variation of φ and H is reduced to the closed variation of $M(\varphi|H)$ with fixed φ and variation of H.

Section I.3

In this section we find conditions for $M^*(\varphi|H(t))$ to be a continuous function of t. Our results on this problem apply only in a much more limited context.

Theorem I.3.1. Let H be a set-valued function from the metric space T to E^n. Suppose that $H(t)$ is closed for each $t \in T$. Let T' be the subset of T consisting of those points t for which $H(t)$ is nonempty.

Let $t_0 \in T'$. Suppose that for every nonnegative convex $\varphi: E^n \to R$ the function

$$t \to M^*(\varphi|H(t))$$

is continuous at t_0 if t_0 is in its domain. Then for every sequence $\{t_m\} \subset T'$ with $t_m \to t_0$,

$$\lim_{m\to\infty} H(t_m) = H(t_0).$$

Proof. Let $\{t_m\} \subset T'$ be a sequence with $t_m \to t_0$. We first show that $\overline{\lim_{m\to\infty}} H(t_m) \subset H(t_0)$. Let $y_0 \in \overline{\lim_{m\to\infty}} H(t_m)$. Then $y_0 \in \underline{\lim_{i\to\infty}} H(t_{m_i})$ for some infinite subsequence $\{m_i\}$ of the positive integers. Applying Theorem I.1.11 with $H_1 = H(t_0)$ and $H_{i+1} = H(t_{m_i})$ for $i > 0$ we obtain a nonnegative convex function $\varphi\colon E^n \to R$ with

$$M^*(\varphi | H(t_{m_i})) = x_i,$$
$$M^*(\varphi | H(t_0)) = x_0,$$

and

$$x_i \to y_0.$$

Since $t \to M^*(\varphi | H(t))$ is continuous at t_0,

$$y_0 = \lim_{i\to\infty} x_i = \lim_{i\to\infty} M^*(\varphi | H(t_{m_i})) = M^*(\varphi | H(t_0)) = x_0 \in H(t_0).$$

Now we show that $H(t_0) \subset \underline{\lim}_{m\to\infty} H(t_m)$. Since $\underline{\lim}_{m\to\infty} H(t_m) \subset \overline{\lim}_{m\to\infty} H(t_m)$ by (I.1.2), this will complete the proof.

Let $y_0 \in H(t_0)$ and apply Theorem I.1.11 with $H_m = H(t_m)$. We obtain a convex function φ with

$$M^*(\varphi | H(t_m)) = x_m$$

and

$$M^*(\varphi | E^n) = y_0.$$

Since $y_0 \in H(t_0)$ and $\varphi(y_0) \leq \varphi(x)$ for all $x \in E^n$, $y_0 = x_0$. Therefore,

$$y_0 = M^*(\varphi|H(t_0)) = \lim_{m\to\infty} M^*(\varphi|H(t_m)) \in \underline{\lim}_{m\to\infty} M(\varphi|H(t_m))$$

$$\subset \underline{\lim}_{m\to\infty} H(t_m). \quad \underline{\text{Q.E.D.}}$$

As the following example shows, the converse of Theorem I.3.1 is false.

<u>Example</u>. Let $T = [0, \infty)$ and $E^n = E^2$. Let

$$H(t) = \begin{cases} \{(t,0),\ (t,\ -1/t)\} & \text{if } t > 0, \\ \{(0,0)\} & \text{if } t = 0. \end{cases}$$

Then $T' = T$ and $\lim_{n\to\infty} H(t_n) = H(t)$ whenever $t_n \to t$. If we define φ: $E^2 \to R$ by

$$\varphi(x_1, x_2) = e^{x_2},$$

then φ is convex and nonnegative and

$$M(\varphi|H(t)) = \begin{cases} \{(t,\ -1/t)\} & \text{if } t > 0, \\ \{(0,0)\} & \text{if } t = 0. \end{cases}$$

Hence $M^*(\varphi|H(t))$ is defined for all $t \in T$, but it is discontinuous at $t = 0$.

We now prove two restricted converses of Theorem I.3.1.

<u>Theorem I.3.2</u>. Let H, T, and T' be as in Theorem I.3.1. Suppose that $H(t)$ is connected for each $t \in T'$. Let $t_0 \in T'$ and suppose that $H(t_0)$ is compact and, for every sequence $\{t_n\} \subset T'$ with $t_n \to t_0$, we have $\lim_{n\to\infty} H(t_n) = H(t_0)$. If $\varphi\colon E^n \to R$ is continuous, then the function

$$t \to M^*(\varphi | H(t))$$

is continuous at t_0 if t_0 is in its domain.

(Note. Theorem I.3.2 does not assume that φ is convex.)

<u>Proof</u>. Suppose t_0 is in the domain of $M^*(\varphi|H)$. Let $\{t_n\}$ be a sequence of points in the domain of $M^*(\varphi|H)$ with $t_n \to t_0$. Then $\{t_n\} \subset T'$, so $\lim_{n\to\infty} H(t_n) = H(t_0)$. Now $H(t_0)$ is compact, so it is contained in a bounded open subset U of E^n. We have $\underline{\lim}_{n\to\infty} H(t_n) = H(t_0) \neq \phi$ and $\overline{\lim}_{n\to\infty} H(t_n) = H(t_0) \subset U$. The boundary of U is a closed bounded subset of E^n, so it is compact. Hence, by Theorem I.1.8, $H(t_n) \subset U$ for n sufficiently large.

The sequence $\{M^*(\varphi|H(t_n))\}$ is bounded, so if it does not converge to $M^*(\varphi|H(t_0))$ then there is an infinite sequence $\{n_i\}$ such that $M^*(\varphi|H(t_{n_i})) \to x_0$, where

$x_0 \notin M^*(\varphi|H(t_0))$. Now $t_{n_i} \to t_0$, so by Theorem I.2.2

$$x_0 \in \varliminf_{i\to\infty} M(\varphi|H(t_{n_i})) \subset M(\varphi|H(t_0)) = \{M^*(\varphi|H(t_0))\}.$$

This is a contradiction. Hence $M^*(\varphi|H(t_n)) \to M^*(\varphi|H(t_0))$. Q.E.D.

Recall that a real-valued function φ on E^n is *quasi*-convex if, for every real a, the set of x for which $\varphi(x) \leq a$ is convex. Every convex function is quasi-convex.

Theorem I.3.3. Let H, T, and T' be as in Theorem I.3.1. Suppose $H(t)$ is closed and convex for every $t \in T$. Let $t_0 \in T'$ and suppose that for every sequence $\{t_m\} \subset T'$ with $t_m \to t_0$, $\lim_{m\to\infty} H(t_m) = H(t_0)$. Let φ be a continuous, quasi-convex function on E^n. If $M(\varphi|H(t_0))$ is nonempty and $M(\varphi|H(t_0)) \subset U$ where U is a bounded open subset of E^n, then $M(\varphi|H(t)) \subset U$ for all t in some neighborhood of t_0.

Proof. Let $M = M(\varphi|H(t_0))$. Now M is closed and bounded so it is compact. Hence, since $M \subset U$, $d(M, E^n - U) = \epsilon > 0$. Let $V = \{x\epsilon E^n | d(M,x) < \epsilon\}$. We have $M \subset V \subset U$ and V is an open bounded subset of E^n.

The set $H(t_0) \cap V \subset H(t_0) \cap \overline{V}$ which is closed, so $\overline{H(t_0) \cap V} \subset H(t_0) \cap \overline{V}$. Let $x \in H(t_0) \cap \overline{V}$. Then $d(x,M) \leq \epsilon$ and, since M is compact, $d(x,M) = d(x,y)$ for some $y \in M \subset H(t_0)$. Let $0 < \lambda < 1$. Then $\lambda y + (1-\lambda)x \in H(t_0)$ because $H(t_0)$ is convex. Furthermore,

$$d(\lambda y + (1-\lambda)x,\ M) \leq d(\lambda y + (1-\lambda)x,\ y) = (1-\lambda)d(x,\ y) < \epsilon,$$

so $\lambda y + (1-\lambda)x \in V$. Therefore,

$$x = \lim_{\lambda \downarrow 0} \lambda y + (1-\lambda)x \in \overline{H(t_0) \cap V}.$$

Hence, $H(t_0) \cap \overline{V} = \overline{H(t_0) \cap V}$.

Suppose the conclusion of Theorem I.3.3 does not hold. Then there is a sequence $\{t_m\} \subset T$ with $t_m \to t_0$ such that $M(\varphi|H(t_m))$ is not contained in U. Hence, $M(\varphi|H(t_m))$ is nonempty, so $\{t_m\} \subset T'$.

Let $\hat{T}$ be the subspace of T' consisting of the points $\{t_m\}$ and the point t_0. Let $\hat{H}$ be the set-valued function from $\hat{T}$ to E^n given by $\hat{H}(t) = H(t) \cap \overline{V}$. If $\{\hat{t}_m\} \subset \hat{T} \subset T'$ is a sequence with $\hat{t}_m \to t_0$, then $\underline{\lim}_{m\to\infty} H(\hat{t}_m) = \lim_{m\to\infty} H(\hat{t}_m) = H(t_0)$. Now $H(t_0) \cap \overline{V} = \overline{H(t_0) \cap V}$, so, by Theorem I.1.10,

$$\underline{\lim}_{m\to\infty} \hat{H}(\hat{t}_m) = \underline{\lim}_{m\to\infty} (H(\hat{t}_m) \cap \overline{V}) = H(t_0) \cap \overline{V} = \hat{H}(t_0).$$

For each m, the set $\hat{H}(t_m)$ is compact, so there is a point $x_m \in M(\varphi|\hat{H}(\hat{t}_m))$. By assumption, there is a point $y_m \in M(\varphi|H(t_m))$ which is not in U and hence not in V. Now $x_m \in \overline{V}$, so the line segment $[x_m, y_m]$ contains a point z_m in the boundary of V. We have $x_m, y_m \in H(t_m)$ which is convex, so $z_m \in H(t_m)$. Therefore, $z_m \in \hat{H}(t_m)$. Now $\varphi(y_m) \leq \varphi(x_m)$ and φ is quasi-convex so $\varphi(z_m) \leq \varphi(x_m)$. Hence, $z_m \in M(\varphi|\hat{H}(t_m))$. There is an infinite sequence $\{m_i\}$ such that $z_{m_i} \to z$ where z is in the boundary of V. By Theorem I.2.2,

$$z = \lim_{i\to\infty} z_{m_i} \in \underline{\lim}_{i\to\infty} M(\varphi|\hat{H}(t_{m_i})) \subset M(\varphi|\hat{H}(t_0)) \subset M(\varphi|H(t_0)).$$

But this contradicts the assumption that $M(\varphi|H(t_0)) \subset V$. Q.E.D.

Corollary I.3.4. Under the hypothesis of Theorem I.3.3, the function

$$t \to M^*(\varphi|H(t))$$

is continuous at t_0 whenever t_0 is in its domain.

Proof. Immediate. Q.E.D.

II

LINEAR CONSTRAINTS

Section II.1

In Part II we study the continuity properties of $M(\varphi|H)$ and $M^*(\varphi|H)$, where φ is a real-valued function defined on E^m, and where H is defined by linear inequalities. In particular, we wish to obtain conditions for $M(\varphi|H)$ to be a closed function and $M^*(\varphi|H)$ to be a continuous function of the parameters in the linear inequalities defining H. According to the results of Part I, we need only look at the behavior of $\underline{\lim}_{i\to\infty} H_i$, and $\overline{\lim}_{i\to\infty} H_i$, where $\{H_i\}$ is a sequence of subsets of Euclidean space, the H_i being defined by linear inequalities whose parameters converge as $i \to \infty$.

Since every linear equality can be represented by two linear inequalities, results for constraint sets defined by systems of linear inequalities can be applied to constraint sets defined by systems of linear equalties and linear inequalities.

We first wish to establish some basic notions and results.

By an <u>affine function</u> $f: E^n \to E^m$ we mean a function definable by $f(x) = ax + b$, where a is an m by n matrix and b is an m-dimensional column vector. Note that f defines a and b uniquely. The functions $f(x)_1, \ldots, f(x)_m$ will be called the <u>coordinates</u> of f.

The numbers $\{a_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\{b_r | 1 \leq r \leq m\}$ will be called the <u>coefficients</u> of f. Every affine function f, from E^n to E^m, can be identified with the point in $E^{(n+1)m}$ defined by its coefficients. Thus, the set of affine functions from E^n to E^m can be regarded as a metric space.

If $x, y \in E^n$, then the statement $x \leq y$ will mean that $x_i \leq y_i$ for each pair of corresponding components x_i, y_i. By $x < y$ we mean that $x_i < y_i$ for each i.

<u>Proposition II.1.1</u>. Let $\{f^r\}$ and f be affine functions from E^n to E^m. The following statements are equivalent:

(<u>a</u>) $f^r \to f$,

(<u>b</u>) $f^r(x) \to f(x)$ for each $x \in E^n$,

(<u>c</u>) $f^r(x) \to f(x)$ uniformly in x on every bounded subset of E^n.

<u>Proof</u>. Immediate. <u>Q.E.D</u>.

If f is an affine function from E^n to E^m, then $H(f)$ will denote the subset of E^n defined by

$$H(f) = \{x \in E^n | f(x) \leq 0\}.$$

We will also consider the set function

$$f \to H(f) \cap C,$$

where C is a fixed subset of E^n. This set function will be denoted $H \cap C$.

We are thus interested in the behavior of such objects as $\underline{\lim}_{r\to\infty} (H(f^r) \cap C)$ and $\overline{\lim}_{r\to\infty} (H(f^r) \cap C)$ for convergent sequences, $\{f^r\}$, of affine functions.

<u>Theorem</u>. Let C be a closed convex set with nonempty (topological) interior. Let f and $\{f^r\}$ be affine functions from E^n to E^m with $f^r \to f$. Then

(II.1.2) $\overline{\lim}_{r\to\infty} (H(f^r) \cap C) \subset H(f) \cap C$,

(II.1.3) $\underline{\lim}_{r\to\infty} (H(f^r) \cap C)$ is a closed convex subset of $H(f) \cap C$,

(II.1.4) if $H(f) \cap C$ has nonempty interior and no component of f is identically zero, then $\lim_{r\to\infty} (H(f^r) \cap C = H(f) \cap C$, and

(II.1.5) if $H(f) \cap C$ has empty interior or some component of f is identically zero, then, for any closed convex subset Q of $H(f) \cap C$, the functions f^r may be chosen so that

$$\underline{\lim}_{r\to\infty} (H(f^r) \cap C) = Q.$$

<u>Proof of (II.1.2)</u>. Let $x \in \overline{\lim}_{r\to\infty} H(f^r) \cap C)$. Then $x = \lim x_i$, where $x_i \in H(f^{r_i}) \cap C$ for some infinite sequence $\{r_i\}$. Now C is closed, so $x \in C$. The sequence $\{x_i\}$ is bounded because it is convergent and $f^{r_i} \to f$ as $i \to \infty$. Hence,

$$f(x) = \lim_{i\to\infty} f^{r_i}(x_i) \leq 0,$$

so $x \in H(f)$. <u>Q.E.D</u>. (II.1.2).

(II.1.3) follows from (II.1.2) and properties (I.1.2), (I.1.4), and (I.1.7) of the inner limit. <u>Q.E.D</u>. (II.1.3).

The proof of (II.1.4) and (II.1.5) depends on the following lemmas.

<u>Lemma II.1.6</u>. A closed convex subset Q of E^n is equal to the intersection of countably many closed half-spaces.

<u>Proof</u>. According to Berge ([1], p. 166), Q is representable in the form $Q = \bigcap_{\alpha\in A} H_\alpha$ where $\alpha \in A$ indexes the supporting half-spaces of Q. Since E^n has a countable basis, A has a countable subset A' with

$$\bigcap_{\alpha\in A'} H_\alpha = \bigcap_{\alpha\in A} H_\alpha. \quad \underline{\text{Q.E.D}}. \text{ (II.1.6)}.$$

<u>Lemma II.1.7</u>. The closure of the interior of a convex set Q with nonempty interior is closure of Q.

<u>Proof</u>. Eggleston ([2], Corollary 3, p. 11). <u>Q.E.D</u>. (II.1.7).

Now suppose that the hypotheses of (II.1.4) are satisfied. Let x be in the interior of $H(f) \cap C$. We claim $f(x) < 0$. If not, $f_i(x) = 0$ for some component f_i of f. Now f_i is nonconstant, for if it were constant it would be identically zero. Hence, there are points y arbitrarily close to x for which $f_i(y) > 0$. This contradicts the assumption that x is in the interior of $H(f) \cap C$.

Now,

$$\lim_{r\to\infty} f^r(x) = f(x) < 0,$$

so $f^r(x) < 0$ for r sufficiently large. Hence, $x \in H(f^r) \cap C$ for r sufficiently large. Therefore,

$$x = \lim_{r\to\infty} x \in \underline{\lim}_{r\to\infty} (H(f^r) \cap C).$$

Since $\underline{\lim}_{r\to\infty} (H(f^r) \cap C)$ is closed and contains the interior of $H(f) \cap C$, by Lemma II.1.7, $H(f) \cap C \subset \underline{\lim}_{r\to\infty} H(f^r) \cap C$. Hence,

$$\underline{\lim}_{r\to\infty} (H(f^r) \cap C) \subset \overline{\lim}_{r\to\infty} (H(f^r) \cap C) \subset H(f) \cap C \subset \underline{\lim}_{r\to\infty} H(f^r) \cap C,$$

and the conclusion follows. Q.E.D. (II.1.4).

Finally, suppose that the hypotheses of (II.1.5) are satisfied. Suppose that $f(x) < 0$ for some $x \in C$. Then no component of f vanishes indentically, so $H(f) \cap C$ must have empty interior. But there is an open neighborhood U of x such that $f(y) < 0$ for $y \in U$. By Lemma II.1.7, U intersects the interior of C so a nonempty open subset of U is contained in $H(f) \cap C$. This is a contradiction. Hence, the system of inequalities $f(x) < 0$ has no solution $x \in C$. Therefore, according to Berge ([1], p. 200), there is a point $\theta \in E^m$ with $\theta \geq 0$ and $\theta \neq 0$ such that

$$\theta \cdot f(x) = \sum_{i=1}^{m} \theta_i f_i(x) \geq 0 \quad \text{for all } x \in C.$$

By Lemma II.1.6, there is a sequence of affine functions $\{g^1, g^2, \ldots\}$, $g^r\colon E^n \to R$, such that Q consists of exactly those points x for which $g^r(x) \leq 0$ for all r. Since $g^r(x) \leq 0$ if and only if $\lambda g^r(x) \leq 0$, where λ is any positive real number, we may assume that the coefficients of the g^r are uniformly bounded.

We define a sequence of affine functions $\{h^r\}$, $h^r\colon E^n \to R$ as follows:

$$h^1 = g^1$$
$$h^2 = 1/2\ g^1,\ h^3 = 1/3\ g^2,$$
$$h^4 = 1/4\ g^1,\ h^5 = 1/5\ g^2,\ h^6 = 1/6\ g^3,$$
$$\vdots$$

Let i_0, $1 \leq i_0 \leq m$, be such that $\theta_{i_0} > 0$. Let $f^r\colon E^n \to E^m$ be the affine function with components

$$f_j^r \begin{cases} f_j & \text{if } j \neq i_0, \\ f_{i_0} + h^r & \text{if } j = i_0. \end{cases}$$

Then $f^r \to f$ because the coefficients of the g^r are uniformly bounded.

Let $x \in Q \subset H(f) \cap C$. Then $f(x) \leq 0$ and $g^r(x) \leq 0$ for all r. Hence, $f^r(x) \leq 0$ for all r, so

$$x = \lim_{r\to\infty} x \in \underline{\lim}_{r\to\infty} (H(f^r) \cap C).$$

Therefore, $Q \subset \underline{\lim}_{r\to\infty} (H(f^r) \cap C)$.

Now let $x \in \overline{\lim}_{r\to\infty} (H(f^r) \cap C)$. Then $x = \lim_{r\to\infty} x_r$, where $x_r \in H(f^r) \cap C$ for r sufficiently large. If $x \notin Q$, then $g^k(x) > 0$ for some k. Hence, $g^k(x_r) > 0$ for r sufficiently large. Let $\{r_i\}$ be the infinite sequence of integers such that $h^{r_i} = 1/r_i\ g^k$. For i sufficiently large, $h^{r_i}(x_{r_i}) = 1/r_i g^k(x_{r_i}) > 0$. Now $x_{r_i} \in H(f^{r_i}) \cap C \subset C$, so $\theta \cdot f(x_{r_i}) \geq 0$. Hence,

$$0 < \theta_{i_0} h^{r_i}(x_{r_i}) \leq \theta_{i_0} h^{r_i}(x_{r_i}) + \theta \cdot f(x_{r_i}) = \theta \cdot f^{r_i}(x_{r_i}),$$

but this contradicts the fact that $f^{r_i}(x_{r_i}) \leq 0$. Therefore, $x \in Q$. Q.E.D. (II.1.5).

We will call an affine function f nondegenerate with respect to the set C if f satisfies the hypothesis of (II.1.4). Otherwise, f will be said to be degenerate with respect to C. Theorems II.1.2 through II.1.5, together with the preceding theorems, show that if φ is continuous and C is closed and convex, then the set function $M(\varphi | H(f) \cap C)$ of f is closed at every nondegenerate point f (see Theorem I.2.2). Furthermore, $M^*(\varphi | H(f) \cap C)$ is continuous at a nondegenerate point f in its domain, provided that either φ is quasi-convex or $H(f) \cap C$ is bounded (see Theorems I.3.2 and I.3.3).

On the other hand, if f is degenerate, then for some continuous φ, $M(\varphi | H \cap C)$ is not closed at f, and for some convex φ, f is in the domain of $M^*(\varphi | H \cap C)$ but this function is not continuous at f.

Section II.2

Even though the functions $M(\varphi|H \cap C)$ and $M^*(\varphi|H \cap C)$ need not be well behaved at a degenerate point when we consider them as functions on the entire space of affine functions, we can guarantee that the functions are respectively closed and continuous at a degenerate point f if we restrict their domains to a suitable subspace of the space of affine functions. To make our results as sensitive as possible, we will phrase them in terms of conditions on a sequence $\{f^r\}$ of affine functions converging to an affine function f which will insure that $\underline{\lim}_{r\to\infty} (H(f^r) \cap C)$ and $\lim_{r\to\infty} (H(f^r) \cap C)$ have suitable properties even though f is degenerate.

Our basic results are Theorems II.2.1 and II.2.2. These theorems are basic only in the logical sense. The most useful results are their corollaries, which are given in Section II.3. The corollaries have, generally, the same conclusions; but have hypotheses which are easier to verify in practice.

The proofs of II.2.1 and II.2.2 are long and complicated. We suggest that anyone planning to read them first arm himself with motivation by reading Sec. II.3.

Theorem II.2.2 asserts that under certain conditions either $\lim_{r\to\infty} H(f^r) = H(f)$ or $H(f^r)$ is empty for infinitely many r. This is equivalent to the condition on the limiting behavior of the $H(f^r)$ hypothesized by Theorems

I.3.2 and I.3.3. Furthermore it implies that $\underline{\lim}_{r\to\infty} H(f^r)$ is either empty or equal to $H(f)$, which is the condition hypothosized by Theorem I.2.2.

We have included Theorem II.2.3 for completeness. It asserts that Theorem II.2.2 is in some sense a "best possible" result.

Let

$$P^m = \{\theta \in E^m | \theta \geq 0 \text{ and } \theta \neq 0\}.$$

If $\theta \in P^m$, we define the <u>carrier of</u> θ by

$$\text{carr } \theta = \{i | 1 \leq i \leq m, \theta_i > 0\}.$$

Let

$$\Delta^{m-1} = \{\theta \in P^m | \theta_1 + \ldots + \theta_m = 1\}.$$

Then Δ^{m-1} is a compact subset of P^m.

<u>Theorem II.2.1.</u> Let $\{f^r\}$ and f be affine functions from E^n to E^m with $f^r \to f$. Let C be a closed convex subset of E^n. Suppose that, for each $\theta \in \Delta^{m-1}$ such that $\theta \cdot f(x) \geq 0$ for all $x \in C$, there is a sequence $\{\theta^r\} \subset \Delta^{m-1}$ such that carr $\theta^r \subset$ carr θ and $\theta^r \cdot f^r(x) \leq 0$ for all $x \in C$ and all sufficiently large r. Then

$$\lim_{r\to\infty} (H(f^r) \cap C) = H(f) \cap C.$$

<u>Proof</u>. By Theorem II.1.2.it is sufficient to show that $H(f) \cap C \subset \underline{\lim}_{r\to\infty} (H(f^r) \cap C)$. Suppose that $x_0 \in H(f) \cap C$ and $x_0 \notin \underline{\lim}_{r\to\infty} (H(f^r) \cap C)$. Then there is an $\epsilon > 0$ and an infinite subset I of the positive integers such that $d(x_0, H(f^r) \cap C) \geq \epsilon$ for each $r \in I$. Hence, if we let

$$N = \{x \in C | d(x, x_0) \leq \tfrac{\epsilon}{2}\},$$

the system of inequalities

$$f^r(x) \leq 0$$

has no solution in N when $r \in I$. Now N is a compact convex subset of E^n so, according to Berge ([1], p. 202), for each $r \in I$ there is a point $\theta^r \in \Delta^{m-1}$ such that

$$\theta^r \cdot f^r(x) > 0, \quad \text{for } x \in N.$$

We may assume that θ^r is chosen so that carr θ^r contains as few elements as possible.

Since each carr θ^r is a subset of a fixed finite set, there is an infinite subset I' of I such that carr θ^r = carr θ^s for every $r, s \in I'$. Since Δ^{m-1} is compact, there is an infinite subset $J \subset I'$ such that the sequence $\{\theta^r\}$, $r \in J$ converges to a point $\theta \in \Delta^{m-1}$.

The points $\{\theta^r\}$, $r \in J$ all have the same carrier, so carr $\theta \subset$ carr θ^r for $r \in J$. Furthermore, since $f^r \to f$,

$$\theta \cdot f(x) \geq 0 \quad \text{for } x \in N.$$

Now $x_0 \in H(f) \cap C \subset H(f)$ and $x_0 \in N$, so

$$0 \geq \theta \cdot f(x_0) \geq 0$$

so that $\theta \cdot f(x_0) = 0$. Let x be any point in C. For each real t let

$$g(t) = \theta \cdot f((1-t)x_0 + tx).$$

We have $g(0) = \theta \cdot f(x_0) = 0$. Since x_0 and x are both in C, $(1-t)x_0 + tx \in C$ for $0 \leq t \leq 1$. Hence, by the definition of N, $(1-t)x_0 + tx \in N$ for all sufficiently small positive t. Thus, for all small positive t,

$$g(t) = \theta \cdot f(1-t)x_0 + tx) \geq 0.$$

This, the linearity of g, and the fact that $g(0) = 0$ imply that $\theta \cdot f(x) = g(1) \geq 0$. We have thus shown that

$$\theta \cdot f(x) \geq 0 \quad \text{for all } x \in C.$$

Let r be a sufficiently large element of J. Since $\theta \in \Delta^{m-1}$, by hypothesis there is a point $\varphi^r \in \Delta^{m-1}$ such that $\text{carr } \varphi^r \subset \text{carr } \theta \subset \text{carr } \theta^r$ and

$$\varphi^r \cdot f^r(x) \leq 0 \quad \text{for all } x \in C.$$

Hence,

$$(\theta^r - t\varphi^r) \cdot f^r(x) > 0, \quad \text{for } x \in N \text{ and } t \geq 0.$$

Since $\text{carr } \varphi^r \subset \text{carr } \theta^r$, we can choose $t > 0$ so that $\theta^r - t\varphi^r \geq 0$ and $\text{carr } (\theta^r - t\varphi^r) \underset{\neq}{\subset} \text{carr } \theta^r$. Now $\theta^r - t\varphi^r \neq 0$ because $x_0 \in N$ and hence $(\theta^r - t\varphi^r) \cdot f^r(x_0) > 0$. Therefore, there is a $\lambda > 0$ such that $\lambda\theta^r - \lambda t\varphi^r \in \Delta^m$. But

$$(\lambda\theta^r - \lambda t\varphi^r) \cdot f^r(x) > 0 \quad \text{for} \quad x \in N$$

and carr $(\lambda\theta^r - \lambda t\varphi^r)$ = carr $(\theta^r - t\varphi^r) \subsetneq$ carr θ^r which contradicts the minimality of the carrier of θ^r. Q.E.D.

We will now use Theorem II.2.1 to prove a more general result. First, we need some additional notation.

Let f be an affine function from E^n to E^m. The (unique) matrix, a, such that $f(x) = ax + b$ will be called the <u>matrix of f</u> and the <u>rank of</u> this matrix will be called the <u>rank of f</u>. If

$$I = \{i_1, \ldots, i_r\}$$

is a subset of $\{1, \ldots, m\}$, then E^I will denote the space of r-tuples of real numbers $(x_{i_1}, \ldots, x_{i_r})$ indexed on the set I. f_I will denote the affine function from E^n to E^I with coordinates $(f_{i_1}, \ldots, f_{i_r})$. If c is a point in E^m, then $f + c$ will denote the affine function with coordinates

$$f_i + c_i, \qquad 1 \leq i \leq m.$$

<u>Theorem II.2.2</u>. Let $\{f^r\}$ and f be affine functions from E^n to E^m with $f^r \to f$. Let

$$I = \{i \mid 1 \leq i \leq m \text{ and } f_i(x) = 0 \text{ for all } x \in H(f)\}.$$

Suppose that $\limsup_{r\to\infty} (\text{rank } f_I^r) \leq \text{rank } f_I$. Then either

$$\lim_{r\to\infty} H(f^r) = H(f)$$

or $H(f^r)$ is empty for infinitely many r in which case

$$\underline{\lim}_{r\to\infty} H(f^r)$$

is empty.

<u>Proof</u>. If $H(f)$ is empty, the conclusion follows from Theorem II.1.2, taking $C = E^n$, so we suppose that $H(f)$ is nonempty.

<u>Lemma II.2.3</u>. Let $J = \{1, 2, \ldots, m\} - I$. There is an $x_0 \in H(f)$ such that $f_J(x_0) < 0$.

<u>Proof</u>. By definition, for each $j \in J$ there is an $x_j \in H(f)$ such that $f_j(x_j) < 0$. Let x_0 be the average of the x_j, $j \in J$. Q.E.D. (II.2.3).

<u>Lemma II.2.4.</u> There is a $\theta \in E^I$ such that $\theta > 0$ and $\theta \cdot f_I$ is identically zero.

<u>Proof</u>. For each $i \in I$ we will construct a $\theta \in E^I$ with $\theta \geq 0$, $\theta_i > 0$, and $\theta \cdot f_I = 0$. The sum of these

θ's will be the point required.

Let $i_0 \in I$ and suppose that the system of inequalitie

$$f_i(x) \leq 0, \quad i \in I, \quad i \neq i_0,$$
$$f_{i_0}(x) + 1 \leq 0$$

has a solution $x \in E^n$. Since $f_J(x_0) < 0$, for t sufficiently small and positive, $(1 - t)x_0 + tx \in H(f)$. But $x_0 \in H(f)$ so $f_{i_0}(x_0) \leq 0$. This and the fact that $f_{i_0}(x) \leq -1$ imply that

$$f_{i_0}((1 - t)x_0 + tx) < 0,$$

contradicting the fact that $i_0 \in I$. Hence, the system of inequalities has no solution. By Motzkin's Theorem there exists $\theta \in E^I$ with $\theta \geq 0$ such that

$$\theta \cdot f_I(x) + \theta_{i_0} > 0 \quad \text{for all} \quad x \in E^n.$$

Hence, $\theta \cdot f_I$ is constant. Now $\theta \cdot f_I(x_0) = 0$ (because $x_0 \in H(f)$) so $\theta \cdot f_I = 0$. Finally,

$$\theta_{i_0} = \theta_{i_0} + \theta \cdot f_I(x_0) > 0.$$

Q.E.D. (II.2.4).

If $H(f^r)$ is empty for infinitely many r, we are done. Hence, we may assume that there is a sequence $\{x_r\} \subset E^n$ with $x_r \in H(f^r)$ for r sufficiently large.

Let $k = \text{rank } f_I$ and let I' be a subset of I containing exactly k elements such that $\text{rank } f_{I'} = k$. Since the rank of a matrix is a lower semicontinuous function of its coefficients,

$$k = \text{rank } f_{I'} \leq \liminf_{r\to\infty} \text{rank } f^r_{I'} .$$

By assumption,

$$\limsup_{r\to\infty} \text{rank } f^r_I \leq \text{rank } f_I = k .$$

Hence, for r sufficiently large,

$$k \leq \text{rank } f^r_{I'} \leq \text{rank } f^r_I \leq k,$$

so that $\text{rank } f^r_{I'} = \text{rank } f^r_I = k$, for all large r.

Let $K = \{0\} \cup I' \cup J$. Define affine functions h and $\{h^r\}$ from E^n to E^K by

$$h_i(x) = \begin{cases} -\sum_{j\in I'} f_j(x) & \text{if } i = 0, \\ f_i(x) & \text{if } i \in I' \cup J, \end{cases}$$

$$h_i^r(x) = \begin{cases} -\sum_{j\in I'} (f_j^r(x) - f_j(x_r)) & \text{if } i = 0, \\ f_i^r(x) - f_i^r(x_r) & \text{if } i \in I', \\ f_i^r(x) & \text{if } i \in J. \end{cases}$$

These functions have the following properties:

(1) $H(f) \subset H(h)$,

(2) $H(h^r) \subset H(f^r)$ for r sufficiently large, and

(3) $h^r \to h$.

Statement (1) is trivial since, if $x \in H(f)$, then $f_J(x) \leq 0$ and $f_I(x) = 0$. To see (2), let r be so large that $x_r \in H(f^r)$ and $\operatorname{rank} f_{I'}^r = \operatorname{rank} f_I^r = k$. Let $x \in H(h^r)$ and let $i \in I'$. Then

$$0 \geq h_i^r(x) \geq \sum_{i\in I'} h_i^r(x) = -h_0^r(x) \geq 0,$$

so $f_{I'}^r(x) = f_{I'}^r(x_r)$. If i is any element of I, then

$$f_i^r = \mu \cdot f_{I'}^r + C$$

for some $\mu \in E^{I'}$ and some constant C. Hence,

$$f_i^r(x) = \mu \cdot f_{I'}^r(x) + C = \mu \cdot f_{I'}^r(x_r) + C = f_i^r(x_r).$$

Therefore, $f_I^r(x) = f_I^r(x_r)$. But $x_r \in H(f^r)$, so $f_I^r(x_r) \leq 0$. Thus,

$$f_I^r(x) \leq 0.$$

Finally,

$$f_J^r(x) = h_J^r(x) \leq 0,$$

so $x \in H(f^r)$.

To prove (3) it suffices to show that

$$f_I^r(x_r) \to 0.$$

Lemma II.2.5. For r sufficiently large, there is a $\theta^r \in E^I$ such that $\theta^r \cdot f_I^r$ is constant. The sequence $\{\theta^r\}$ may be chosen so that $\theta^r \to \theta$ (where θ is as in Lemma II.2.4), in which case $\theta^r > 0$ for r sufficiently large (because $\theta > 0$).

Proof. Let r be so large that rank f_I^r = rank $f_{I'}^r$. Then there is a $\mu^r \in E^{I'}$ and a constant c^r such that

$$\theta \cdot f_I^r = \mu^r \cdot f_{I'}^r + c^r.$$

Now μ^r is the unique solution of a system of linear equations whose coefficients depend continuously on f_I^r, so μ^r is a continuous function of f_I^r. In the limit,

$$\theta \cdot f_I = 0 = 0 \cdot f_{I'} + 0,$$

so $\mu^r \to 0$. Define θ^r by

$$\theta_i^r = \begin{cases} \theta_i - \mu_i^r & \text{if } i \in I', \\ \theta_i & \text{if } i \in I - I'. \end{cases}$$

Then $\theta^r \to \theta$ and

$$\theta^r \cdot f_I^r = \theta \cdot f_I^r - \mu^r \cdot f_{I'}^r = c^r.$$

<u>Q.E.D.</u> (II.2.5)

Now let r be so large that $x_r \in H(f^r)$ and $\theta^r > 0$. Let $i \in I$. Then

$$0 \geq \theta_i^r \, f_i^r(x_r) \geq \theta^r \cdot f_I^r(x_r) = \theta^r \cdot f_I^r(x_0).$$

Now $\theta^r \cdot f_I^r(x_0) \to \theta \cdot f_I(x_0) = 0$ and $\theta_i^r \to \theta_i \neq 0$. Therefore, $f_i^r(x_r) \to 0$, and (3) is established.

We claim that the functions h and $\{h^r\}$ satisfy the hypotheses of Theorem II.2.1 with $C = E^n$. Let ℓ be the cardinality of K. Suppose that for some $\varphi \in \Delta^{\ell-1}$,

$\varphi \cdot h(x) \geq 0$ for all $x \in E^n$. If $j \in J$, $h_j(x_0) = f_j(x_0) < 0$. But $x_0 \in H(f) \subset H(h)$, so

$$\varphi_j \ h_j(x_0) \geq \varphi \cdot h(x_0) \geq 0.$$

Therefore, $\varphi_j = 0$. Hence,

$$\varphi \cdot h = \varphi_0 h_0 + \sum_{i \in I'} \varphi_i h_i = \sum_{i \in I'} (\varphi_i - \varphi_0) f_i.$$

Now $\varphi \cdot h$ is constant and the rows of the matrix of $f_{I'}$ are linearly independent, so

$$\varphi_i - \varphi_0 = 0 \quad \text{for each} \quad i \in I'.$$

Consequently, for each r,

$$\varphi \cdot h^r = \varphi_0 h_0^r + \sum_{i \in I'} \varphi_i h_i^r = \sum_{i \in I'} (\varphi_i - \varphi_0) h_i^r = 0 \leq 0,$$

so we may take the sequence $\{\varphi^r\}$ required by the hypotheses of Theorem II.2.1 to be constantly equal to φ.

Now we have, by (1), (2), and Theorem II.2.1,

$$H(f) \subset H(h) = \lim_{r \to \infty} H(h^r) = \underline{\lim}_{r \to \infty} H(h^r) \subset \underline{\lim}_{r \to \infty} H(f^r),$$

so Theorem II.2.2 follows from Theorem II.1.2. Q.E.D. (II.2.2).

In the above argument we have tacitly assumed that I and J are nonempty. If I is empty, the result reduces to Theorem II.1.4. If J is empty, we take x_0 to be any point in $H(f)$, and the above argument applies.

Observe that the hypotheses of Theorem II.2.2 place restrictions only on the matrices of the functions f_I^r, not their constant coefficients. We now show that Theorem II.2.2 is the "best possible" result in the sense that if the condition on the matrices of the f_I^r is not satisfied, then for suitable choices of the constant coefficients of the f^r, the conclusion of Theorem II.2.2 does not hold.

<u>Theorem II.2.6</u>. Let f, $\{f^r\}$, and I be as in Theorem II.2.2, with $f^r \to f$. Suppose that $H(f)$ is nonempty and that $\limsup_{r\to\infty} \text{rank } f_I^r > \text{rank } f_I$. Then there is a sequence $\{c^r\}$ of points in E^m with $c_i^r = 0$ for $i \notin I$ and $c^r \to 0$ such that $\underline{\lim}_{r\to\infty} H(f^r + c^r)$ is a proper, nonempty subset of $H(f)$.

<u>Proof.</u> Let $J = \{1, 2, \ldots, m\} - I$. By Lemma II.2.3 there is a point $x_0 \in H(f)$ such that $f_J(x_0) < 0$. Define the sequence c^r by

$$c_i^r \begin{cases} -f_i^r(x_0) & \text{if } i \in I, \\ 0 & \text{if } i \in J. \end{cases}$$

Then $c^r \to 0$ (because $f_I^r(x_0) \to f_I(x_0) = 0$).

We have

$$f_I^r(x_0) + c_I^r = f_I^r(x_0) - f_I^r(x_0) = 0 \leq 0$$

and

$$\lim_{r\to\infty} f_J^r(x_0) = f_J(x_0) < 0,$$

so $x_0 \in H(f^r + c^r)$ for r sufficiently large. Hence, $x_0 \in \underline{\lim}_{r\to\infty} H(f^r + c^r)$.

It remains to show that there is a point in $H(f)$ which is not in $\underline{\lim}_{r\to\infty} H(f^r + c^r)$. Let A denote the matrix of f_I, and A^r denote the matrix of f_I^r. Then

$$\limsup_{r\to\infty} \operatorname{rank} A^r > \operatorname{rank} A,$$

so there is an infinite subset K of the positive integers such that

$$\operatorname{rank} A^r > \operatorname{rank} A \quad \text{for all } r \in K.$$

In the following a point in E^k will be regarded as a $k \times 1$ matrix. If M is a matrix, $\tilde{M}$ will denote its transpose. For $v \in E^k$, $\|v\| = (\tilde{v}v)^{1/2}$ will denote the usual norm in E^k.

Let

$$V = \{v \in E^n | Av = 0\}.$$

For each $r \in K$ let

$$C_r = \{\tilde{A}^r\theta | \theta \in E^I \text{ and } \theta > 0\}.$$

Now V and C_r are convex subsets of E^n. Suppose that for all sufficiently large $r \in K$, the sets V and C_r are disjoint. Then by Berge ([1], p. 163), for each such $r \in K$ there is a hyperplane separating V and C_r. Hence there is nonzero $u^r \in E^n$ and a real number a^r such that

$$u^r \cdot x \geq a^r \quad \text{if } x \in V,$$

and

$$u^r \cdot x \leq a^r \quad \text{if } x \in C_r.$$

We may assume $\|u^r\| = 1$.

If $x \in V$, then $\lambda x \in V$ for any real λ. Therefore,

$$\lambda u^r \cdot x \geq a^r \quad \text{for all real } \lambda \text{ and } x \in V.$$

Hence,

$$u^r \cdot x = 0 \quad \text{for any } x \in V,$$

and also

$$a^r \leq 0.$$

This implies that $\tilde{u}^r$ is a linear combination of the rows of A. That is,

$$\tilde{u}^r = \tilde{\theta}^r A \quad \text{or} \quad u^r = \tilde{A}\theta^r$$

for some $\theta^r \in E^I$. If I' is the set of indices of a maximal linearly independent subset of the rows of A, we may choose θ^r with $\sup \theta^r \subset I'$. With this additional restriction θ^r is unique, and it is a continuous function of u^r.

Let $\{r_i\} \subset K$ be an infinite subsequence such that $u^{r_i} \to u$. Then $\theta^{r_i} \to \theta$, where $u = \tilde{A}\theta$. The coordinates of the points θ^{r_i} are uniformly bounded since $\{\theta^{r_i}\}$ is a convergent subsequence. By Lemma II.2.4 there is a $\overline{\theta} \in E^I$ with $\overline{\theta} > 0$ and $\overline{\theta} \cdot f_I = 0$. For λ sufficiently large, $\theta^{r_i} + \lambda\overline{\theta} > 0$ for all i. Now $\tilde{A}\overline{\theta} = 0$, so

$$\tilde{A}(\theta^{r_i} + \lambda\overline{\theta}) = \tilde{A}\theta^{r_i} = u^{r_i}.$$

We have $\tilde{A}^{r_i}(\theta^{r_i} + \lambda\overline{\theta}) \in C_{r_i}$, so

$$\tilde{u}^{r_i}\, \tilde{A}^{r_i}(\theta^{r_i} + \lambda\overline{\theta}) \leq a^{r_i} \leq 0.$$

Therefore,

$$0 \geq \lim_{i\to\infty} \tilde{u}^{r_i} \tilde{A}^{r_i}(\theta^{r_i} + \lambda\bar{\theta}) = \tilde{u}\,\tilde{A}(\theta + \lambda\bar{\theta}) = \tilde{u}u = 1,$$

which is a contradiction. Consequently, $C_r \cap V$ is nonempty for infinitely many $r \in K$.

If $C_r \cap V = \{0\}$, then $\tilde{A}^r\theta = 0$ for some $\theta \in E^I$ with $\theta > 0$. Therefore,

$$\tilde{A}^r x = \tilde{A}^r(x + \lambda\theta) \in C_r \quad \text{for} \quad x \in E^I$$

if we take λ so large that $x + \lambda\theta > 0$. Hence, C_r is the subspace of E^n generated by the rows of A^r. Now

$$\dim V + \dim C_r = n - \operatorname{rank} A + \operatorname{rank} A^r > n.$$

This contradicts our assumption that $V \cap C_r = \{0\}$. Therefore, for infinitely many $r \in K$ there is a nonzero $v^r \in V \cap C_r$ which we may assume to have norm 1.

Let $\{r_i\} \subset K$ be an infinte sequence such that $v^{r_i} \to v$. Then $\|v\| = 1$, and $Av = 0$. Since $f_J(x_0) < 0$, we may choose $\sigma > 0$ so small that $f_J(x_0 + \sigma v) \leq 0$. Now

$$f_I(x_0 + \sigma v) = f_I(x_0) + A(\sigma v) = f_I(x_0) \leq 0,$$

so $x_o + \sigma v \in H(f)$. Suppose that $x_o + \sigma v \in \underline{\lim}_{r\to\infty} H(f^r + c^r)$. Then $x_o + \sigma v = \lim_{r\to\infty} x_r$, where $x_r \in H(f^r + c^r)$ for r sufficiently large.

For each i, let $u^{r_i} = x_{r_i} - x_0$. Then $u^{r_i} \to \sigma v$. For i sufficiently large,

$$A^{r_i} u^{r_i} = A^{r_i} x_{r_i} - A^{r_i} x_0 = f_I^{r_i}(x_{r_i}) - f_I^{r_i}(x_0)$$

$$= f_I^{r_i}(x_{r_i}) + c_I^{r_i} \leq 0.$$

Now $v^{r_i} \in C_{r_i}$, so $v^{r_i} = \tilde{A}^{r_i} \theta^{r_i}$ for some $\theta^{r_i} > 0$. Therefore,

$$\tilde{v}^{r_i} u^{r_i} = \tilde{\theta}^{r_i} A^{r_i} u^{r_i} \leq 0 \quad \text{for } i \text{ large}.$$

Hence,

$$0 \geq \lim_{i\to\infty} \tilde{v}^{r_i} u^{r_i} = \tilde{v}\sigma v = \sigma > 0.$$

This contradiction shows that $x_0 + \sigma v \notin \underline{\lim}_{r\to\infty} H(f^r + c^r)$. Q.E.D. (II.2.6).

Section II.3

In this section we obtain a number of results which follow quickly from the theorems of Sec. II.2. Some of these results have hypotheses which are considerably easier to verify than those of Sec. II.2.

Corollary II.3.1. Let f be an affine function from E^n to E^m, and let c^r be a sequence of points in E^m with $c^r \to 0$. Then either

$$\lim_{r\to\infty} H(f + c^r) = H(f)$$

or $H(f + c^r)$ is empty for infinitely many r.

Proof. Since the matrix of $f + c^r$ is the same as the matrix of f, this follows at once from Theorem II.2.2. Q.E.D.

Corollary II.3.2. Let f and $\{f^r\}$ be affine functions from E^n to E^m with $f^r \to f$. Let

$$I = \{i | 1 \leq i \leq m \text{ and } f_i(x) = 0 \text{ for all } x \in H(f)$$

If the matrix of f_I has full rank, then either

$$\lim_{r\to\infty} H(f^r) = H(f)$$

or $H(f^r)$ is empty for infinitely many r.

Proof. Theorem II.2.1. Q.E.D.

If f and g are affine functions from E^n to E^m and $E^{m'}$, respectively, we define

$$H(f, g) = \{x \in E^n) \mid f(x) \leq 0 \text{ and } g(x) = 0\}.$$

Thus $H(f, g)$ represents a constraint set defined by a mixed system of linear equalities and linear inequalities.

Corollary II.3.3. Let f, $\{f^r\}$, g, $\{g^r\}$ be affine functions from E^n to E^m and $E^{m'}$ with $f^r \to f$ and $g^r \to g$. Let C be a convex subset of E^n. Suppose for every θ, φ with $\theta \in E^m$, $\varphi \in E^{m'}$, $\theta \geq 0$, for which not both θ and φ are zero and such that $\theta \cdot f(x) \geq 0$ and $\theta \cdot g(x) = 0$ for all $x \in C$, there are sequences $\{\epsilon^r\}$ of E^m and $\{\varphi^r\}$ of $E^{m'}$ such that for all sufficiently large r we have $\theta^r \geq 0$, not both θ^r and φ^r are zero, $\theta^r \cdot f(x) \leq 0$ and $\varphi^r \cdot g(x) = 0$ for all $x \in C$, and carr $\theta^r \subset$ carr θ and carr $\varphi^r \subset$ carr φ. Then

$$\lim_{r\to\infty} (H(f^r, g^r) \cap C) = H(f, g) \cap C.$$

Proof. Let $h = (f, g, -g)$ and $h^r = (f^r, g^r, -g^r)$. Then $h^r \to h$ and $H(h) = H(f, g)$. Apply Theorem II.2.1. Q.E.D.

Corollary II.3.4. Let f, $\{f^r\}$, g, $\{g^r\}$ be affine functions from E^n to E^m and $E^{m'}$ with $f^r \to f$, $g^r \to g$. Let

$$I = \{i | 1 \leq i \leq n,\ f_i(x) = 0 \ \text{ for all } \ x \in H(f, g)\}.$$

Suppose that $\limsup_{r\to\infty} \operatorname{rank} (f_I^r, g^r) \leq \operatorname{rank} (f_I, g)$. Then either $\lim_{r\to\infty} H(f^r, g^r) = H(f, g)$ or $H(f^r, g^r)$ is empty for infinitely many r.

Proof. Let $h = (f, g, -g)$ and $h^r = (f^r, g^r, -g^r)$. Then $h^r \to h$, $H(h) = H(f, g)$, and $H(h^r) = H(f^r, g^r)$. Let

$$I' = \{i | h_i(x) = 0 \ \text{ for all } \ x \in H(h)\}.$$

Then $h_{I'} = (f_I, g, -g)$ and $h_{I'}^r = (f_I^r, g^r, -g^r)$. Now $\operatorname{rank} (f_I, g, -g) = \operatorname{rank} (f_I, g)$ and $\operatorname{rank} (f_I^r, g^r, -g^r) = \operatorname{rank} (f_I^r, g^r)$, so the conclusion follows from Theorem II.2.2. Q.E.D.

Corollary II.3.5. Let f, $\{f^r\}$ and g, $\{g^r\}$ be affine functions from E^n to E^m and $E^{m'}$, respectively, with $f^r \to f$ and $g^r \to g$. Let

$$I = \{i \mid 1 \leq i \leq m \text{ and } f_i(x) = 0 \text{ for all } x \in H(f, g)\}.$$

Let A be the matrix of f_I and B be the matrix of g. If the matrix $\binom{A}{B}$ has full rank, then either

$$\lim_{r\to\infty} H(f^r, g^r) = H(f, g)$$

or $H(f^r, g^r)$ is empty for infinitely many r.

Proof. Corollary II.3.4. Q.E.D.

Corollary II.3.6.* Let g and $\{g^r\}$ be affine functions from E^n to E^m with $g^r \to g$. Let

$$H = \{x \in E^m \mid x \geq 0 \text{ and } g(x) = 0\}.$$

Let

$$H^r = \{x \in E^m \mid x \geq 0 \text{ and } g^r(x) = 0\}.$$

* This result in the case where I is empty and the g^r all have the same matrix was obtained by Lloyd Shapley prior to the authors' proof.

Let

$$I = \{r \mid 1 \leq r \leq n \text{ and } x_r = 0 \text{ for all } x \in H\}.$$

If the matrix obtained from the matrix of g by deleting those columns whose indices are in I has full rank, then either $\lim_{r\to\infty} H^r = H$ or H^r is empty for infinitely many r.

<u>Proof</u>. Corollary II.3.5 and some simple matrix manipulations. Q.E.D.

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DOCUMENT CONTROL DATA

1. ORIGINATING ACTIVITY	2a. REPORT SECURITY CLASSIFICATION
THE RAND CORPORATION	UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE

ON THE CONTINUITY OF THE MINIMUM SET OF A CONTINUOUS FUNCTION

4. AUTHOR(S) (Last name, first name, initial)

Folkman, Jon H., George B. Dantzig, Norman Shapiro

5. REPORT DATE	6a. TOTAL NO. OF PAGES	6b. NO. OF REFS.
February 1966	65	21

7. CONTRACT or GRANT NO.	8. ORIGINATOR'S REPORT NO.
AF 49(638)-1700	RM-4657-PR

9a. AVAILABILITY/LIMITATION NOTICES	9b. SPONSORING AGENCY
DDC 1	United States Air Force Project RAND

10. ABSTRACT

Part of a continuing study of the mathematical and computational aspects of chemical equilibrium theory. The Memorandum obtains necessary and sufficient conditions so the solution of a constrained minimization problem will vary continuously when the constraints and objective function are varied. The study also obtains special results when the constraints are linear inequalities. 65 pp. Bibliog.

11. KEY WORDS

Mathematics
Computer programs
Linear programming
Chemical equilibrium